Classical & Quantum Waves

Lecture 13-1

Transport of energy by a wave

- Last lecture, we saw that a traveling wave contains both kinetic and potential energy.

- Total energy is a portion of a vibrating string between $a \leq x \leq b$:

$$E = \frac{1}{2}\mu \int_a^b dx \left[\underbrace{\left(\frac{\partial y}{\partial t}\right)^2}_{\text{Kinetic}} + \underbrace{v^2 \left(\frac{\partial y}{\partial x}\right)^2}_{\text{Potential}} \right]$$

Kinetic: Energy from string moving up/down in y-direction

Potential: Energy from string extending as wave moves by

* $\frac{\partial y}{\partial t} \neq v$!

velocity of wave is along x-direction!

- For a pulsed wave:

y-displacement = 0 everywhere except for within the pulse

$\rightarrow$ all the energy is inside the pulse (from eq. above)

- For a sinusoidal wave, $y = A\sin(kx - \omega t)$

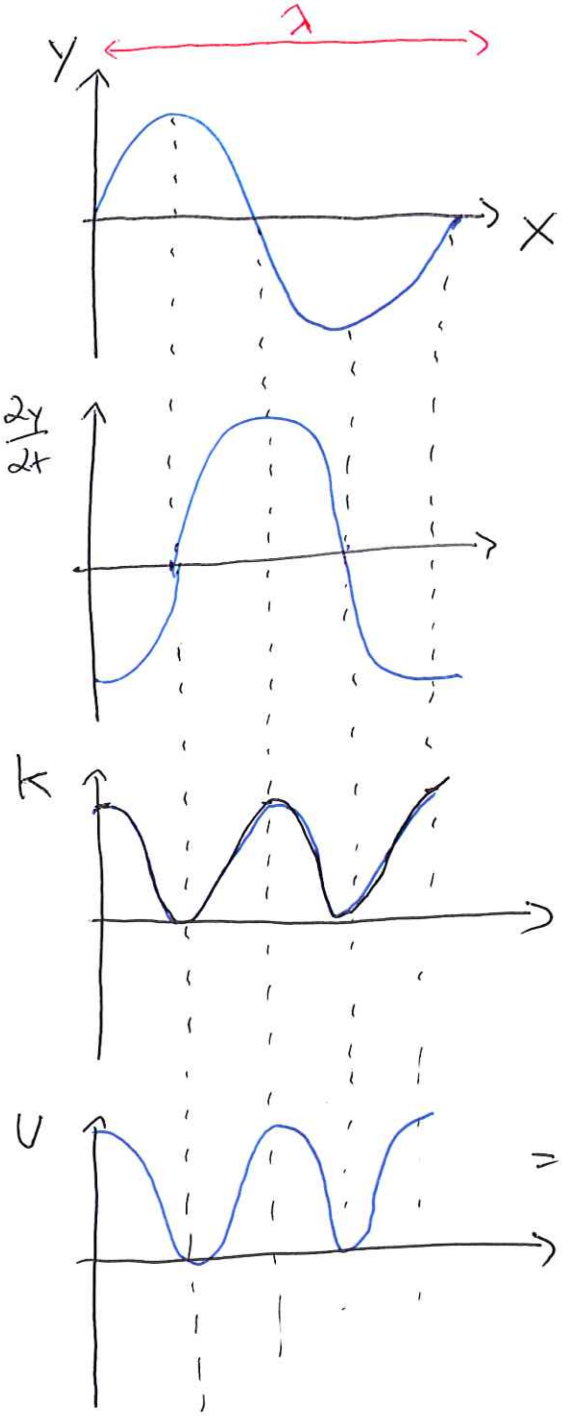
$$\frac{\partial y}{\partial t} = -\omega A \cos(kx - \omega t)$$

- Using above expression for $E = K + U$

$$K = \frac{1}{2}\mu \delta x \omega^2 A^2 \cos^2(kx - \omega t)$$

[refer to previous lecture]

$$U = \frac{1}{2} v^2 \mu \delta x k^2 A^2 \cos^2(kx - \omega t)$$



$\Rightarrow$ max when $\left|\frac{\partial y}{\partial t}\right|$ max ~~[illegible]~~ $K \sim \left(\frac{\partial y}{\partial t}\right)^2$

$\Rightarrow$ max when $\left|\frac{\partial y}{\partial x}\right|$ max $\quad U \sim \left(\frac{\partial y}{\partial x}\right)^2$

$\hookrightarrow$ when $y(x)$ passes through zero (max slope)

$\Rightarrow$ Qualitatively different result from harmonic oscillator! (analogy b/t coupled HOs & waves breaks down here)

For HO, U was max when displacement max b/c had $\frac{1}{2}ky^2$ restoring force $\rightarrow$ we don't have that here.

- Total energy in one λ-segment is $E_{tot} = \frac{1}{2}\mu\omega^2 A^2 \lambda$

[from last lecture]

- The time to travel length λ is $\frac{\lambda}{v}$

- Energy transported per unit time (ie, Power) is then

$$\frac{E_{tot}}{\frac{\lambda}{v}} = \boxed{\text{Power} = \frac{1}{2}\mu\omega^2 A^2 v} \qquad \propto \omega^2, A^2, v$$

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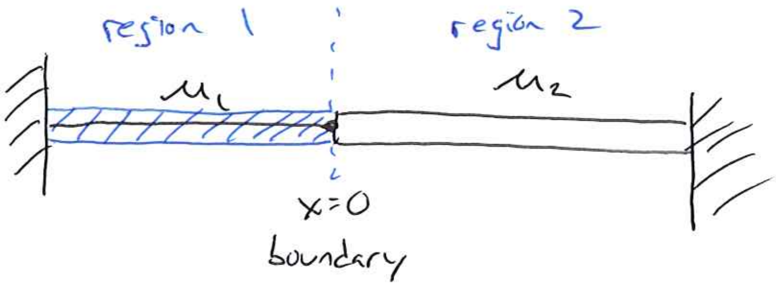
Waves at discontinuities

- When a wave encounters a discontinuity at a boundary between two different media, in general part of the incoming wave will be reflected, and part will be transmitted.

→ Here we determine the amplitudes and phases of these three waves

- Many examples where this occurs:
  - mechanichal vibrations
  - light /optics
  - acoustics
  - quantum
  - ...

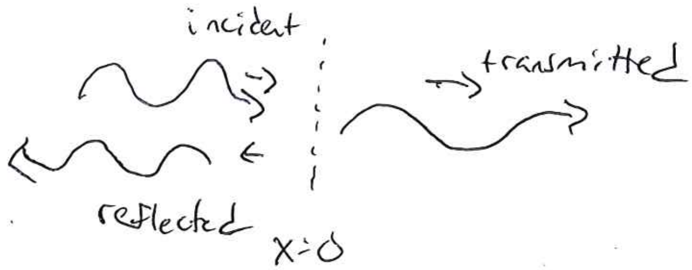
- Consider two strings w/ different mass per unit length μ_1 and μ_2. Connected at $x=0$, held under tension T



$$V_1 = \sqrt{\frac{T}{\mu_1}}$$

$$V_2 = \sqrt{\frac{T}{\mu_2}}$$

- We generate an incident wave in region 1, we want to know the amplitude and phase of reflected and transmitted wave at the boundary b/t two regions at $x=0$



$$y_I = A_1 \cos(\omega t - k_1 x)$$

$$y_T = A_2 \cos(\omega t - k_2 x)$$

$$y_R = B_1 \cos(\omega t + k_1 x)$$

↳ + b/c moving to left

Considerations / boundary conditions

1) At the boundary, where two strings connected, displacements at $x=0$ for both strings must be the same

→ this means that $\underbrace{y_I(x=0) + y_R(x=0)}_{\text{region 1 amplitude at } x=0} = \underbrace{y_T(x=0)}_{\text{region 2 amplitude at } x=0}$

Define $y_1 = y_I + y_R$ (waves in region 1)

(waves in region 2) $y_2 = y_T$ (just to keep notation consistent)

$$\Rightarrow \boxed{y_1(x=0,t) = y_2(x=0,t)}$$

Boundary condition

- This condition also means that the frequencies ω must also be the same on both sides of the boundary

$$\boxed{\omega_1 = \omega_2 = \omega}$$

- However, since wave velocities will be different in two regions, the wavelengths must be different b/c $\lambda = \frac{2\pi}{\omega} v$

$$\Rightarrow \boxed{k_1 = \frac{\omega}{v_1} \neq k_2 = \omega / v_2}$$

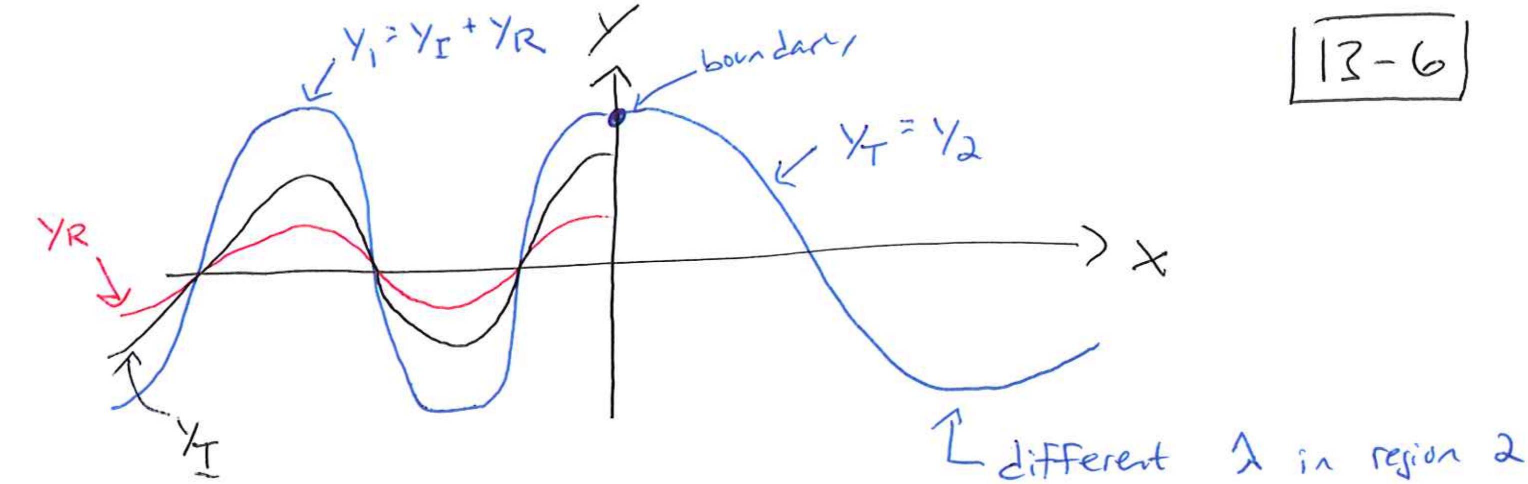
$\left[\lambda = \frac{2\pi}{k}, \quad V = \frac{\omega}{k} = \frac{2\pi f}{\frac{2\pi}{\lambda}} = f\lambda \right]$

Consideration 2)

- Restoring force at boundary must be continuous, otherwise the finite difference in force acting on an infinitely small segment/mass would lead to infinite acceleration, which is unphysical.
   - Transverse Force $= T\left(\frac{\partial y}{\partial x}\right)$ [recall last lecture]

$\Rightarrow$ slopes $\frac{\partial y}{\partial x}$ must be continuous

$$\boxed{\frac{\partial y_1}{\partial x}(x=0,t) = \frac{\partial y_2}{\partial x}(x=0,t)}$$



$$y_1 = y_I + y_R = A_1 \cos(\omega t - k_1 x) + B_1 \cos(\omega t + k_1 x)$$

$$y_2 = y_T = A_2 \cos(\omega t - k_2 x)$$

- Boundary condition (1) gives $y_1 = y_2$ @ $x = 0$

$$\Rightarrow A_1 \cos(\omega t - k_1 x) + B_1 \cos(\omega t + k_1 x) = A_2 \cos(\omega t - k_2 x)$$

@ $x = 0$. Must be true for all times, Evaluate at $t = 0$

$$\Rightarrow \boxed{A_1 + B_1 = A_2}$$

(as in drawing above)

- Boundary condition (2) gives $\frac{dy_1}{dx} = \frac{dy_2}{dx}$ @ $x = 0$

$$\frac{\partial y_1}{\partial x} = \underbrace{-(-k_1)}_{+k_1} A_1 \sin(\omega t - k_1 x) - k_1 B_1 \sin(\omega t + k_1 x)$$

$$\frac{\partial y_2}{\partial x} = \underbrace{-(-k_2)}_{+k_2} A_2 \sin(\omega t - k_2 x)$$

$\Rightarrow k_1 A_1 \sin(\omega t - k_1 x) - k_1 B_1 \sin(\omega t + k_1 x) = k_2 A_2 \sin(\omega t - k_2 x)$

@ $x = 0$. Evaluate at $t = \frac{\pi}{2\omega}$ to make the sin term $= 1$ (we ~~can't~~ can do this b/c must be true for all times)

$$\Rightarrow \boxed{k_1 A_1 - k_1 B_1 = k_2 A_2}$$

- We want the ratios of amplitudes A_2/A_1 for the transmitted and incident waves and B_1/A_1 for the reflected & incident waves.

$\rightarrow B_1 = A_2 - A_1 \qquad (1)$

(2) $\Rightarrow k_1 A_1 - k_1(A_2 - A_1) = k_2 A_2$

$\rightarrow k_1 A_1 + k_1 A_1 - k_1 A_2 = k_2 A_2$

$\rightarrow 2k_1 A_1 = A_2(k_1 + k_2)$

$$\rightarrow \boxed{\frac{A_2}{A_1} = \frac{2k_1}{k_1 + k_2} = T_{12}}$$

"Transmission coefficient" (of amplitude)

$0 < T_{12} < 2$ [Take limits $k_1 \rightarrow 0$, $k_2 \rightarrow 0$]

· Manipulate same equations to find B_1/A_1

$$A_2 = B_1 + A_1$$

$$\rightarrow k_1 A_1 - k_1 B_1 = k_2 (B_1 + A_1) \qquad (2)$$

$$\rightarrow \cancel{k_1 A} \; A_1 (k_1 - k_2) = B_1 (k_1 + k_2)$$

$$\Rightarrow \boxed{\frac{B_1}{A_1} = \frac{k_1 - k_2}{k_1 + k_2} = R_{12}}$$

"Reflection coefficient" (of amplitute)

$-1 < R_{12} < +1$ take limits $\begin{bmatrix} k_1 \rightarrow 0 \\ k_2 \rightarrow 0 \end{bmatrix}$

· One can also show, by manipulating the ~~two~~ transmission & reflection coeff. equations, that

$$\boxed{T_{12} = 1 + R_{12}}$$

[to show this, solve for $(k_1 + k_2)$, plug in the other eq. etc.]

~~· ~~From this~~, ~~we~~ one can show that T_{12} is always positive and between 0 and 2~~

~~· R_{12} is between -1 and $+1$~~

• $R_{12} = \dfrac{k_1 - k_2}{k_1 + k_2}$ shows that the sign of the reflected wave depends on whether k_2 is less than or greater than k_1.

• If; $k_2 < k_1$, ~~reflec~~ $R_{12} = \dfrac{B_1}{A_1}$ is positive & reflected wave is in phase w/ incident wave (see earlier drawing)

$\lambda_2 > \lambda_1$

$v_2 > v_1$

$\mu_2 < \mu_1$

If $k_2 > k_1$, $R_{12} = \dfrac{B_1}{A_1}$ is negative & reflected wave is opposite sign (π out of phase) compared to incident wave

• $T_{12} = \dfrac{A_2}{A_1}$ shows that transmitted wave always positive and in phase with incident wave

- Rewrite T_{12} & R_{12} in terms of μ

- $v = \frac{\omega}{k} = \sqrt{\frac{T}{\mu}} \Rightarrow k = \omega\sqrt{\frac{\mu}{T}}$

$$\Rightarrow T_{12} = \frac{2k_1}{k_1 + k_2} = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}} \quad \text{[other terms cancel]}$$

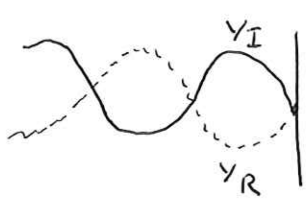
$$R_{12} = \frac{k_1 - k_2}{k_1 + k_2} = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}}$$

- As $\mu_2 \to \infty$, region 2 becomes a rigid wall.

$T_{12} \to 0$ recall $T_{12} = 1 + R_{12}$

no transmitted wave $\Rightarrow R_{12} = -1$

$\longrightarrow$ reflected wave at rigid wall reverses amplitude by getting π phase shift



$Y_T = 0$

$\to$ This concept is universal!

- light / EM waves
- sound waves
- Quantum waves

Waves in two- & three-dimensions

- Description of waves in one dimension can be extended to 2D and 3D

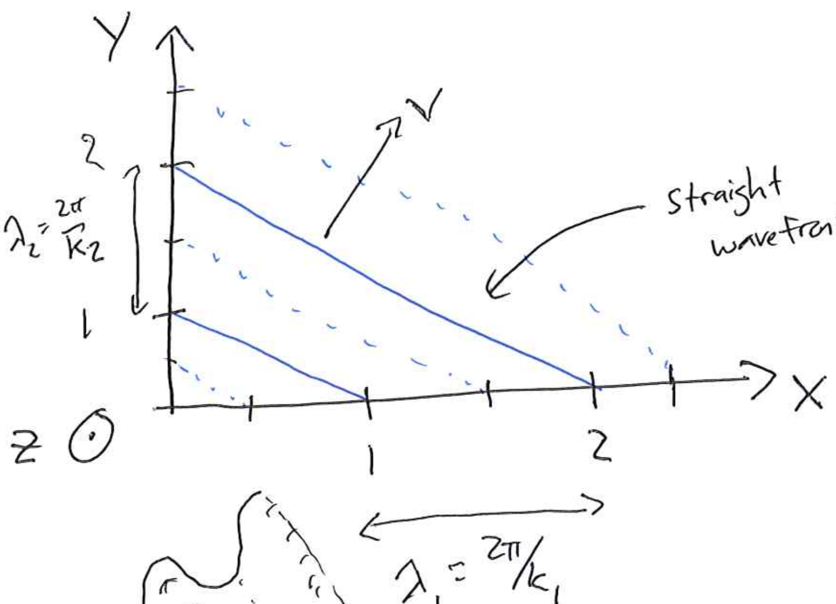
2D wave eq: $$\boxed{\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}}$$ displacement in z-direction

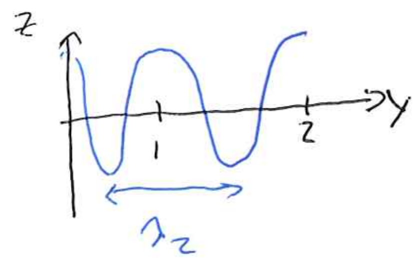
Solution: $$\boxed{z(x,y,t) = A\cos(k_1 x + k_2 y - \omega t)}$$ propagating in x, y direction

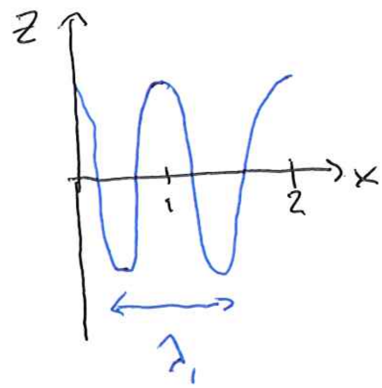
velocity: $$\boxed{v = \frac{\omega}{\sqrt{k_1^2+k_2^2}} = \frac{\omega}{k}}$$

$k = \sqrt{k_1^2+k_2^2}$

Snapshot of wave fronts:







[Attempted 3D illustration...]

2D w/ circular symmetry

$$\boxed{\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}}$$

(no proof)

[pebble in a pond]

Solutions to this are called Bessel functions

→ for $r \to \infty$: $\boxed{\frac{\partial^2 z}{\partial r^2} = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}}$

→ solved by 1D sine wave

$$z(r,t) = A\cos(kr - \omega t)$$

3-dimensions

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}}$$

$= \Delta\psi$ w/ Δ = Laplace operator

Solution: 3D sine wave $\cancel{\partial^2\psi}$ $\psi(x,y,z,t) = \cancel{A\cos(k_1x + k_2x k_3x - \omega t)}$ $A\cos(k_1x + k_2y + k_3z - \omega t)$

$$v = \frac{\omega}{k} \quad \text{w/ } k = \sqrt{k_1^2 + k_2^2 + k_3^2}$$

3D (spherical symmetry)

3D spherical wave eq: $\frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r}\frac{\partial \psi}{\partial r} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}$

Solution: 3D spherical waves: $\psi(r,t) = \underbrace{\frac{A}{r}}_{\text{Amplitude decreases as } \frac{1}{r}} \cos(\omega t - kr)$

$\rightarrow$ Energy $\sim$ amplitude^2 $\propto \frac{1}{r^2}$

- Higher dimension waves relevant to acoustic, EM, quantum... waves!